Influence of anisotropy on anomalous scaling of a passive scalar advected by the Navier-Stokes velocity field

E. Jurčišinová, M. Jurčišin, and R. Remecký
Institute of Experimental Physics, Slovak Academy of Sciences, Watsonova 47, 040 01 Košice, Slovakia
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The influence of weak uniaxial small-scale anisotropy on the stability of the scaling regime and on the anomalous scaling of the single-time structure functions of a passive scalar advected by the velocity field governed by the stochastic Navier-Stokes equation is investigated by the field theoretic renormalization group and operator-product expansion within one-loop approximation of a perturbation theory. The explicit analytical expressions for coordinates of the corresponding fixed point of the renormalization-group equations as functions of anisotropy parameters are found, the stability of the three-dimensional Kolmogorov-like scaling regime is demonstrated, and the dependence of the borderline dimension $d_c \in (2,3]$ between stable and unstable scaling regimes is found as a function of the anisotropy parameters. The dependence of the turbulent Prandtl number on the anisotropy parameters is also briefly discussed. The influence of weak small-scale anisotropy on the anomalous scaling of the structure functions of a passive scalar field is studied by the operator-product expansion and their explicit dependence on the anisotropy parameters is present. It is shown that the anomalous dimensions of the structure functions, which are the same (universal) for the Kraichnan model, for the model with finite time correlations of the velocity field, and for the model with the advection by the velocity field driven by the stochastic Navier-Stokes equation in the isotropic case, can be distinguished by the assumption of the presence of the small-scale anisotropy in the systems even within one-loop approximation. The corresponding comparison of the anisotropic anomalous dimensions for the present model with that obtained within the Kraichnan rapid-change model is done.

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I. INTRODUCTION

Although, theoretical understanding of intermittency and anomalous scaling in fully developed turbulence on the microscopic level remains one of the last unsolved problems of the classical physics [1–4], nevertheless, during the last two decades a great progress has been achieved in the understanding of anomalous scaling of single-time correlation or structure functions of passively advected scalar or vector fields in the framework of models with a given Gaussian statistics of the velocity field. The reason for their investigation is twofold. On one hand, the problems of the passive advection of scalar or vector fields are noticeably easier for the theoretical investigation than the original problem of anomalous scaling of the velocity field driven by the stochastic Navier-Stokes equation and, on the other hand, it was also shown that the deviations from the classical Kolmogorov-Obukhov (KO) theory are even more strongly noticeable for passively advected scalar or vector field than for the velocity field itself (see, e.g., Refs. [4–13]). Such deviations, referred to as anomalous or nondimensional scaling, manifest themselves in a singular dependence of the correlation or structure functions on the distances and on the integral (external) turbulence scale, and it is believed that this phenomenon is related to strong fluctuations of the energy flux.

The crucial role in the studies of passive advection was played by the simple model of a passive scalar quantity advected by a random Gaussian velocity field, white in time and self-similar in space, the so-called Kraichnan rapid-change model [14]. Namely, in the framework of the rapid-change model, for the first time, the anomalous scaling was established on the basis of a microscopic model [15] and corresponding anomalous exponents were calculated within controlled approximations [16,17] (see also survey paper [18] and references cited therein). In Refs. [16,17], the so-called zero-mode approach was applied and it was shown that nontrivial anomalous behavior is related to the zero modes (homogeneous solutions) of the closed system of exact differential equations satisfied by the equal-time correlation functions.

A great progress in the understanding of anomalous scaling in turbulence was also done by the renormalization-group (RG) technique which represents an effective method for the investigation of self-similar scaling behavior [19–21]. In Refs. [22,23], the field theoretic RG and the operator-product expansion (OPE) were used in the systematic investigation of the Kraichnan rapid-change model. It was shown that within the field theoretic RG approach the anomalous scaling is related to the existence in the model of the composite operators with negative critical dimensions in the OPE, which are usually termed as *dangerous* operators (see, e.g., Refs. [21,24,25] for details).

Afterward, the field theoretic RG technique was also used for the investigation of the anomalous behavior of various descendants of the Kraichnan model, namely, models with the inclusion of small-scale anisotropy [26], the compressibility [27–29], the finite correlation time of velocity field [30–34], and the helicity [35]. Besides, advection of the passive vector field by the Gaussian self-similar velocity field (with and without large- and small-scale anisotropies, pressure, compressibility, and finite correlation time) has been also investigated and all possible asymptotic scaling regimes and crossover among them have been classified and anomalous scaling was analyzed [36]. A general conclusion of all these investigations is that the anomalous scaling, which is the most intriguing and important feature of the Kraichnan rapid-change model, remains valid for all generalized models.

Nevertheless, it must be stressed that although the models of turbulent advection by the so-called "synthetic" velocity field describe many features of real turbulent advection they still remain only as toy models of real turbulent systems. This fact has important consequences. On one hand, they are not able to describe some interesting properties of real turbulence, e.g., within the rapid-change model the possible effects of helicity of the system (spatial parity violation) are invisible in the framework of the field theoretic approach [35] and, on the other hand, the models with Gaussian velocity field with finite correlation time, which are able to describe more properties of real turbulence, have their own drawbacks. For example, their Galilean noninvariance [7] leads to the fact that they do not take into account the self-advection of turbulent eddies and, as a result of these the so-called "sweeping effects," the different time correlations of the Eulerian velocity are not self-similar and depend strongly on the integral scale; see, e.g., Ref. [37] (see also Ref. [38]). Therefore, without a doubt, the crucial point for further progress in understanding of properties of turbulent flows at the microscopic level is to go beyond the Gaussianity of the velocity field.

However, the transition to models with more realistic non-Gaussian statistics of the velocity field, e.g., to the models with statistics of the velocity field driven by the stochastic Navier-Stokes equation, obviously leads to mathematical complications in their theoretical investigations, especially when some breaking of a symmetry of the turbulent environment is considered (e.g., anisotropy of energy pumping to the system, compressibility of the fluid, or mirror symmetry violation). This fact is the main reason of the present situation that a lot of studies of the anomalous scaling of a single-time structure functions of a passive scalar advected by a Gaussian velocity field exist (as was briefly discussed above) and that the investigation of these problems within the models with the Navier-Stokes velocity field was done only in the simplest isotropic and incompressible situation [39].

On the other hand, the effects of a symmetry breaking can be interesting. For example, the influence of anisotropy on inertial range behavior of passively advected fields [17,30,31,40–45], as well as the velocity field itself [46–48] is important (see also the survey paper [49] and references cited therein, as well as recent astrophysical investigations, e.g., in Refs. [50,51]). In this respect, it was shown that for the even structure (or correlation) functions the exponents which describe the inertial range scaling behavior exhibit universality and they are ordered hierarchically with respect to the degree of anisotropy with leading contribution given by the exponent from the isotropic shell but, on the other hand, the survival of the anisotropy in the inertial range is demonstrated by the behavior of the odd structure functions, namely, for example, the so-called skewness factor decreases down the scales slower than expected earlier in accordance with the classical KO theory.

However, as was already mentioned, all anisotropic studies of anomalous scaling were done only within toy models of the turbulent advection with a given Gaussian statistics of the velocity field. The main aim of the present paper is to go beyond Gaussian statistics of the velocity field and to study the influence of the so-called small-scale uniaxial anisotropy (see the next section) on the anomalous scaling of the single-time structure functions of a passive scalar field advected by the velocity field driven by the stochastic Navier-Stokes equation, i.e., within the model which takes into account the correlations of the velocity field of higher order and is close to real turbulent advection. But, as was shown in Ref. [52], the models of fully developed turbulence based on the stochastic Navier-Stokes equation with the presence of the small-scale anisotropy are rather difficult for investigations even at one-loop level. Therefore, for simplicity, in what follows, we shall suppose that the corresponding parameters of anisotropy are close to zero, i.e., we shall work in the so-called weak anisotropy limit. As we shall see, such an assumption, on one hand, gives basic information about the behavior of the corresponding anomalous exponents as continuous functions of the anisotropy parameters and, on the other hand, the model is sufficiently easier for mathematical analysis of the problem. The corresponding model with no restrictions on anisotropy parameters is much more complicated and will be analyzed elsewhere.

The main conclusion of the paper is the following: the anomalous scaling of the single-time structure functions of a passive scalar, which is universal (up to needed normalizations) for all models with a Gaussian statistics of the velocity field (the rapid-change model, model with frozen velocity field, etc.) and with a non-Gaussian statistics of the velocity field (model with the velocity field driven by the stochastic Navier-Stokes equation) within the one-loop approximation in the isotropic case (or in the case with the large-scale anisotropy), is strongly nonuniversal when the corresponding models with the small-scale uniaxial anisotropy are considered. In this case, the corresponding anomalous dimensions smoothly depend on the anisotropy parameters of the model.

In the end, let us describe briefly the solution of the problem in the framework of the field theoretic approach [21,24,25]. It can be divided into two main stages. On the first stage the multiplicative renormalizability of the corresponding field theoretic model is demonstrated and the differential RG equations for its correlation functions are obtained. The asymptotic behavior of the latter on their ultraviolet (UV) argument (r/ℓ) for $r \gg \ell$ and any fixed (r/L) is given by infrared (IR) stable fixed points of those equations. Here, ℓ and L are inner (ultraviolet) and outer (infrared) scales (lengths). It involves some "scaling functions" of the infrared argument (r/L), whose form is not determined by the RG equations. On the second stage, the behavior of scaling functions at $r \ll L$ is found from the OPE within the framework of the general solution of the RG equations. There, the crucial role is played by the critical dimensions of various composite operators, which give rise to an infinite family of independent aforementioned scaling exponents (and hence to multiscaling).

The paper is organized as follows. In Sec. II, the field theoretic formulation of the stochastic model is given and the corresponding Feynman diagrammatic technique is briefly discussed. In Sec. III, we perform the UV renormalization of the model, the renormalization constants are calculated in one-loop approximation, and the corresponding RG equations are derived. In Sec. IV, we discuss the stability of the scaling regime of the model, which is governed by the corresponding IR fixed point of the RG equations. In Sec. V, the influence of anisotropy on turbulent Prandtl number is briefly discussed. In Sec. VI, the renormalization of the needed composite operators is done and their critical dimensions are found as functions of parameters of the model. Besides, the anomalous scaling of the single-time structure functions under the influence of anisotropy is discussed. In Sec. VII, the comparison of the obtained results to the corresponding results obtained within the rapid-change model is done. The main results are reviewed and discussed in Sec. VIII.

II. FIELD THEORETIC FORMULATION OF THE MODEL

The advection of a passive scalar field $\theta(x)\equiv\theta(t,\mathbf{x})$ (concentration of an impurity, temperature, etc.) by an incompressible velocity field $\mathbf{v}(x)\equiv\mathbf{v}(t,\mathbf{x})$ ($\partial_i v_i=0$) is described by the system of stochastic equations,

$$\partial_t\theta+(\mathbf{v}\cdot\partial)\theta=\nu_0 u_0\Delta\theta+f^\theta, \tag{1}$$

$$\partial_t\mathbf{v}+(\mathbf{v}\cdot\partial)\mathbf{v}=\nu_0\Delta\mathbf{v}-\partial P+\mathbf{f}^{\mathbf{v}}, \tag{2}$$

where Eq. (1) represents the advection-diffusion equation for the scalar field and Eq. (2) is the stochastic Navier-Stokes equation for the transverse (due to incompressibility) velocity field. In Eqs. (1) and (2) we have used the following standard notation: $\partial_t\equiv\partial/\partial t$; $\partial_i\equiv\partial/\partial x_i$; $\Delta\equiv\partial^2$ is the Laplace operator; ν_0 is the kinematic viscosity (in what follows, the subscript 0 denotes bare parameters of the unrenormalized theory); $\nu_0 u_0$ represents the molecular diffusivity, where the dimensionless reciprocal Prandtl number u_0 is extracted explicitly; $P(x)\equiv P(t,\mathbf{x})$ is the pressure; and the summation over dummy indices is implied. The random noise $f^\theta=f^\theta(x)$ is taken to have a Gaussian distribution with correlator

$$D^\theta(x;x')=\langle f^\theta(x)f^\theta(x')\rangle=\delta(t-t')C(\mathbf{r}/L),\quad \mathbf{r}=\mathbf{x}-\mathbf{x}', \tag{3}$$

where, in what follows, unimportant function $C(\mathbf{r}/L)$ must only decrease rapidly enough for $r\equiv|\mathbf{r}|\gg L$ for some integral scale L. The main role of noise (3) is to preserve the steady state of the system. On the other hand, the explicit form of the transverse random force per unit mass $\mathbf{f}^{\mathbf{v}}$ is essential. Standardly, we assume that it also obeys a Gaussian distribution with zero mean and correlator

$$D^v_{ij}(x;x')=\langle f^{\mathbf{v}}_i(x)f^{\mathbf{v}}_j(x')\rangle =\delta(t-t')\int\frac{d^d\mathbf{k}}{(2\pi)^d}R_{ij}(\mathbf{k})d_f(k)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \tag{4}$$

where d is the dimension of the $\mathbf{x}$ space and $R_{ij}(\mathbf{k})$ is a transverse projector. It describes geometric properties of the random force and, in the simplest isotropic case, is defined as the ordinary transverse projector, namely, $R_{ij}(\mathbf{k})\equiv P_{ij}(\mathbf{k})=\delta_{ij}-k_ik_j/k^2$. The energy pumping function $d_f(k)$ is chosen in such a way to realize a realistic, i.e., infrared introduction (by large-scale eddies) of the energy into the system and, at the same time, it is important that the function $d_f(k)$ must have a power-law asymptotic form at large k. The last condition is necessary for the application of the field theoretic RG technique. Both conditions are satisfied by the function [21,24,25]

$$d_f(k)=D_0k^{4-d-2\varepsilon}, \tag{5}$$

with a positive amplitude $D_0>0$ and the exponent $0<\varepsilon\leq 2$. The most realistic value is $\varepsilon=2$ [21,25]. In Eq. (4), the needed infrared regularization is given by a restriction of the integration from below, namely, $k\geq m$, where m corresponds to another integral scale. We shall suppose that $L\gg 1/m$. For further convenience it is also useful to introduce bare coupling constant g_0 instead of D_0 by the following definition:

$$D_0\equiv g_0\nu_0^3. \tag{6}$$

In addition, g_0 is a formal small parameter of the ordinary perturbation theory, and it is related to the characteristic UV momentum scale Λ (or inner length $l\sim\Lambda^{-1}$) by the relation

$$g_0\simeq\Lambda^{2\varepsilon}. \tag{7}$$

As was already mentioned, the geometric properties of the energy pumping are given by the form of the transverse projector $R_{ij}(\mathbf{k})$. In the present work we shall study the model when the random force has uniaxial anisotropic properties at all scales defined by a unit vector $\mathbf{n}$ (the so-called small-scale uniaxial anisotropy). For this purpose it is convenient to take the transverse projector $R_{ij}(\mathbf{k})$ in the following form:

$$R_{ij}(\mathbf{k})=\left(1+\alpha_1\frac{(\mathbf{n}\cdot\mathbf{k})^2}{k^2}\right)P_{ij}(\mathbf{k})+\alpha_2P_{is}(\mathbf{k})n_sn_tP_{tj}(\mathbf{k}), \tag{8}$$

which is the simplest special case of a general uniaxial anisotropic transverse tensor structure (see, e.g., Ref. [26]). Here, n_i is the ith component of the unit vector $\mathbf{n}$ and the necessary condition for positively defined correlator (4) is that the anisotropy parameters α_1 and α_2 must satisfy inequalities $\alpha_1>-1$ and $\alpha_2>-1$.

Using the general theorem [53], the stochastic model (1)–(4) can be rewritten into the equivalent field theoretic model of the set of fields $\Phi=\{\mathbf{v},\theta,\mathbf{v}',\theta'\}$ with the following action functional:

$$\begin{aligned}S(\Phi)=&\frac{1}{2}\int dt_1d^d\mathbf{x}_1dt_2d^d\mathbf{x}_2[v_i'(t_1,\mathbf{x}_1)D^v_{ij}(t_1,\mathbf{x}_1;t_2,\mathbf{x}_2)v_j'(t_2,\mathbf{x}_2)\\&+\theta'(t_1,\mathbf{x}_1)D^\theta(t_1,\mathbf{x}_1;t_2,\mathbf{x}_2)\theta'(t_2,\mathbf{x}_2)]\\&+\int dt\,d^d\mathbf{x}(\theta'\{-\partial_t-\mathbf{v}\cdot\partial+\nu_0u_0[\Delta+\tau_0(\mathbf{n}\cdot\partial)^2]\}\theta\\&+\mathbf{v}'\{-\partial_t-\mathbf{v}\cdot\partial+\nu_0[\Delta+\chi_{10}(\mathbf{n}\cdot\partial)^2]\}\mathbf{v}\\&+\nu_0\mathbf{n}\cdot\mathbf{v}'[\chi_{20}\Delta+\chi_{30}(\mathbf{n}\cdot\partial)^2]\mathbf{n}\cdot\mathbf{v}),\end{aligned} \tag{9}$$

where $\mathbf{v}'(x)$ and $\theta'(x)$ are the needed auxiliary fields with the same tensor properties as fields $\mathbf{v}(x)$ and $\theta(x)$. D^θ and D^v are the correlation functions given in Eqs. (3) and (4) for the random forces f^θ and $\mathbf{f}^{\mathbf{v}}$, respectively. The terms with new unrenormalized parameters τ_0, χ_{10}, χ_{20}, and χ_{30}, which are not present in the original stochastic equations (1) and (2), are related to the presence of small-scale uniaxial anisotropy and they must be introduced into the action to make the model multiplicatively renormalizable (see, e.g., Refs. [26,52] for details).

Because the auxiliary vector field $\mathbf{v}'(x)$ is also transverse, i.e., $\partial_i v_i'=0$, it allows one to omit the pressure term in Eq. (9): the corresponding term has the following form:

$$\int dt\, d^d\mathbf{x} v_i' \partial_i P,$$

and after the integration by parts it is evident that it vanishes, namely,

$$\int dt\, d^d\mathbf{x} v_i' \partial_i P = -\int dt\, d^d\mathbf{x} P \partial_i v_i' = 0.$$

Model (9) corresponds to a standard Feynman diagrammatic technique. Explicit analytical expressions for, in what follows, important bare propagators $\langle\theta'\theta\rangle_0$, $\langle v_i' v_j\rangle_0$, and $\langle v_i v_j\rangle_0$ are (in the frequency-momentum representation)

$$\langle\theta'\theta\rangle_0 = \frac{1}{i\omega + \nu_0 u_0 k^2 + \tau_0 \nu_0 u_0 (\mathbf{n}\cdot\mathbf{k})^2}, \tag{10}$$

$$\langle v_i' v_j\rangle_0 = \frac{P_{ij}}{K} - \frac{\widetilde{K} P_{is} n_s n_t P_{tj}}{K[K+\widetilde{K}(1-\xi_k^2)]}, \tag{11}$$

$$\langle v_i v_j\rangle_0 = -\frac{K_1 P_{ij}}{KK^*} + \frac{P_{is} n_s n_t P_{tj}}{K[K^* + \widetilde{K}(1-\xi_k^2)]}\left[\frac{\widetilde{K}K_1}{K^*} + \frac{\widetilde{K}[K_1 + K_2(1-\xi_k^2)]}{K+\widetilde{K}(1-\xi_k^2)} - K_2\right], \tag{12}$$

where

$$K = i\omega + \nu_0 k^2 + \nu_0 \chi_{10} (\mathbf{n}\cdot\mathbf{k})^2, \tag{13}$$

$$K^* = -i\omega + \nu_0 k^2 + \nu_0 \chi_{10} (\mathbf{n}\cdot\mathbf{k})^2, \tag{14}$$

$$K_1 = -g_0 \nu_0^3 k^{4-d-2\varepsilon}(1+\alpha_1 \xi_k^2), \tag{15}$$

$$K_2 = -g_0 \nu_0^3 k^{4-d-2\varepsilon}\alpha_2, \tag{16}$$

$$\widetilde{K} = \nu_0 \chi_{20} k^2 + \nu_0 \chi_{30} (\mathbf{n}\cdot\mathbf{k})^2, \tag{17}$$

$$\xi_k^2 = \frac{(\mathbf{n}\cdot\mathbf{k})^2}{k^2}. \tag{18}$$

Propagators (10)–(12) are written in the form suitable also for the investigation of the strong small-scale anisotropy case, i.e., when no restrictions on the anisotropy parameters α_1 and α_2 are imposed. In our case, when weak anisotropy is supposed, it is enough to work with linear parts of the propagators with respect to all anisotropy parameters, but for shortness we shall not present their explicit forms here. The graphical representation of propagators (10)–(12) is shown in Fig. 1 (the ends with a slash in the propagators $\langle\theta'\theta\rangle_0$ and $\langle v'v\rangle_0$ correspond to the fields θ' and v', respectively, and the ends without a slash correspond to the fields θ and v, respectively). The triple vertices (or interaction vertices) $-\theta' v_j \partial_j \theta = \theta' v_j V_j \theta$ and $-v_i' v_j \partial_j v_l = v_i' v_j W_{ijl} v_l/2$, where $V_j = ik_j$ and $W_{ijl} = i(k_l \delta_{ij} + k_j \delta_{il})$ (in the momentum-frequency representation), are present in Fig. 2, where momentum $\mathbf{k}$ is flowing into the vertices via the auxiliary fields θ' and v', respectively.

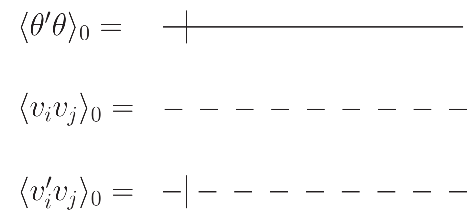

FIG. 1. Graphical representation of needed propagators of the model.

It is well known that the formulation of the problem through the action functional (9) replaces the statistical averages of random quantities in the stochastic problem defined by Eqs. (1)–(4) with equivalent functional averages with weight $\exp S(\Phi)$. The generating functionals of the total Green's functions $G(A)$ and the connected Green's functions $W(A)$ are then defined by the functional integral [21]

$$G(A) = e^{W(A)} = \int \mathcal{D}\Phi\, e^{S(\Phi)+A\Phi}, \tag{19}$$

where $A(x)=\{A^\theta, A^{\theta'}, \mathbf{A}^{\mathbf{v}}, \mathbf{A}^{\mathbf{v}'}\}$ represents a set of arbitrary sources for the set of fields Φ, $\mathcal{D}\Phi \equiv \mathcal{D}\theta\mathcal{D}\theta'\mathcal{D}\mathbf{v}\mathcal{D}\mathbf{v}'$ denotes the measure of functional integration, and the linear form $A\Phi$ is standardly defined as

$$A\Phi = \int dx[A^\theta(x)\theta(x) + A^{\theta'}(x)\theta'(x) + A_i^v(x) v_i(x) + A_i^{v'}(x) v_i'(x)]. \tag{20}$$

III. RENORMALIZATION-GROUP ANALYSIS

Using the standard field-theoretic analysis of canonical dimensions leads to the information about possible UV divergences in the model (see, e.g., Refs. [20,21]). Due to the fact that the dynamical model (9) belongs to the class of the so-called two-scale models [21,24,25], the canonical dimension of some quantity F is given by two numbers, namely, the momentum dimension d_F^k and the frequency dimension

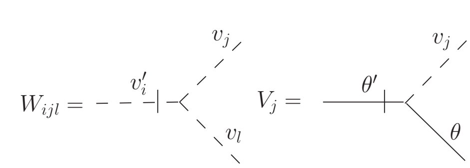

FIG. 2. The triple (interaction) vertices of the model. Momentum $\mathbf{k}$ is flowing into the vertices via the auxiliary fields θ' and v'.

TABLE I. Canonical dimensions of the fields and parameters of the model under consideration.

F	$\mathbf{v}$	$\mathbf{v}'$	θ	θ'	m,Λ,μ	ν_0,ν	g_0	g,u_0,u,τ_0,τ χ_{i0},χ_i $(i=1,2,3)$
d_F^k	-1	$d+1$	0	d	1	-2	2ε	0
d_F^ω	1	-1	$-1/2$	$1/2$	0	1	0	0
d_F	1	$d-1$	-1	$d+1$	1	0	2ε	0

d_F^ω. To find the dimensions of all quantities, it is convenient to use the standard normalization conditions $d_k^k=-d_x^k=1$, $d_\omega^\omega=-d_t^\omega=1$, $d_k^\omega=d_x^\omega=d_\omega^k=d_t^k=0$, and the requirement that each term of the action functional must be dimensionless separately with respect to the momentum and frequency dimensions. The total canonical dimension d_F is then defined as $d_F=d_F^k+2d_F^\omega$ [it is related to the fact that $\partial_t \propto \nu_0\partial^2$ in the free action (9) with the choice of zero canonical dimension for ν_0]. In the framework of the theory of renormalization the total canonical dimension in dynamical models plays the same role as the momentum dimension does in static models. The canonical dimensions of our model are present in Table I, where also the canonical dimensions of the renormalized parameters are shown.

The field theoretic model (9) is logarithmic at $\varepsilon=0$ (the coupling constant g_0 is dimensionless); therefore, in the framework of the minimal subtraction (MS) scheme [20], which is always used in what follows, possible UV divergences in the correlation functions have the form of poles in parameter ε. The superficial divergences can be present only in the one-irreducible Green's functions for which the corresponding total canonical dimensions are a non-negative integer. Detailed analysis of the possible divergences of the model was done, e.g., in Ref. [39]; therefore, it is not used to repeat it here. This analysis shows that superficially divergent functions of our model are only functions $\langle v'v\rangle_{1-ir}$ and $\langle\theta'\theta\rangle_{1-ir}$ and action (9) has all necessary tensor structures to remove divergences multiplicatively. It can be explicitly expressed in the multiplicative renormalization of the parameters g_0, u_0, ν_0, τ_0, and χ_{i0} $(i=1,2,3)$ in the following form:

$$\nu_0=\nu Z_\nu,\quad g_0=g\mu^{2\varepsilon}Z_g,\quad u_0=uZ_u, \tag{21}$$

$$\tau_0=\tau Z_\tau,\quad \chi_{i0}=\chi_i Z_{\chi_i}, \tag{22}$$

with $i=1,2,3$. Here, the dimensionless parameters g, u, ν, τ, and χ_i are the renormalized counterparts of the corresponding bare ones; μ is the renormalization mass (a scale setting parameter), an artifact of the dimensional regularization. Quantities $Z_i=Z_i(g,u,\tau,\chi_i;d;\varepsilon)$ are the so-called renormalization constants. They contain poles in ε.

The renormalized action functional has the following form:

$$S_R(\Phi)=\frac{1}{2}\int dt_1d^d\mathbf{x}_1dt_2d^d\mathbf{x}_2[v_i'(t_1,\mathbf{x}_1)D_{ij}^v(t_1,\mathbf{x}_1;t_2,\mathbf{x}_2)v_j'(t_2,\mathbf{x}_2)+\theta'(t_1,\mathbf{x}_1)D^\theta(t_1,\mathbf{x}_1;t_2,\mathbf{x}_2)\theta'(t_2,\mathbf{x}_2)]$$
$$+\int dt\, d^d\mathbf{x}(\theta'\{-\partial_t-\mathbf{v}\cdot\partial+\nu u[Z_5\Delta+\tau Z_6(\mathbf{n}\cdot\partial)^2]\}\theta+\mathbf{v}'\{-\partial_t-\mathbf{v}\cdot\partial+\nu[Z_1\Delta+\chi_1Z_2(\mathbf{n}\cdot\partial)^2]\}$$
$$\times\mathbf{v}+\nu\mathbf{n}\cdot\mathbf{v}'[\chi_2Z_3\Delta+\chi_3Z_4(\mathbf{n}\cdot\partial)^2]\mathbf{n}\cdot\mathbf{v}). \tag{23}$$

By comparison of the renormalized action (23) with definitions of the renormalization constants Z_j, where $j=g,u,\nu,\tau,\chi_i$ $(i=1,2,3)$, which are given in Eqs. (21) and (22), one comes to the relations among them as follows:

$$Z_\nu=Z_1,\quad Z_g=Z_1^{-3},\quad Z_u=Z_5Z_1^{-1},$$

$$Z_{\chi_i}=Z_{i+1}Z_1^{-1},\quad Z_\tau=Z_6Z_5^{-1}. \tag{24}$$

The renormalization constants Z_i $(i=1,\ldots,6)$ are determined by the requirement that the one-irreducible Green's functions $\langle v'v\rangle_{1-ir}$ and $\langle\theta'\theta\rangle_{1-ir}$ must be UV finite when are written in the renormalized variables, i.e., they have no singularities in the limit $\varepsilon\to 0$. On the other hand, the one-irreducible Green's functions $\langle v'v\rangle_{1-ir}$ and $\langle\theta'\theta\rangle_{1-ir}$ are related to the corresponding self-energy operators $\Sigma^{v'v}$ and $\Sigma^{\theta'\theta}$, which are expressed via Feynman diagrams, by the corresponding Dyson equations. In the frequency-momentum representation they can be written in the following convenient form:

$$\langle v_i'v_j\rangle_{1-ir}=[-i\omega+\nu_0p^2+\nu_0\chi_{10}(\mathbf{n}\cdot\mathbf{p})^2]\delta_{ij}+[\nu_0\chi_{20}p^2$$
$$+\nu_0\chi_{30}(\mathbf{n}\cdot\mathbf{p})^2]n_in_j-\Sigma_{ij}^{v'v}(\omega,p), \tag{25}$$

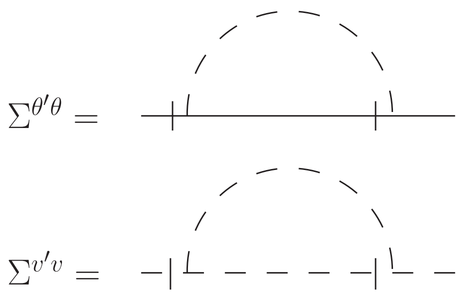

FIG. 3. The one-loop diagrams that contribute to the self-energy operators $\Sigma^{\theta'\theta}$ and $\Sigma^{v'v}$.

$$\langle \theta' \theta \rangle_{1-ir} = -i\omega + \nu_0 u_0 p^2 + \nu_0 u_0 \tau_0 (\mathbf{n}\cdot\mathbf{p})^2 - \Sigma_{\theta'\theta}(\omega,p). \tag{26}$$

Thus, Z_i ($i=1,\ldots,6$) are found from the requirement that the UV divergences are canceled in Eqs. (25) and (26) after the substitution $e_0 = e\mu^{d_e} Z_e$. This determines Z_i ($i=1,\ldots,6$) up to an UV finite contribution, which is fixed by the choice of the renormalization scheme. In the MS scheme all the renormalization constants have the form 1+ *poles in* ε. In one-loop approximation the self-energy operators $\Sigma^{v'v}$ and $\Sigma^{\theta'\theta}$ are given by Feynman diagrams, which are shown in Fig. 3, and their explicit analytical form is given as follows:

$$\Sigma_{ij}^{v'v}(p) = -\frac{S_d}{(2\pi)^d}\frac{g\nu}{2\varepsilon}[p^2\delta_{ij}A_1 + (\mathbf{n}\cdot\mathbf{p})^2\delta_{ij}A_2 + p^2 n_i n_j A_3 + (\mathbf{n}\cdot\mathbf{p})^2 n_i n_j A_4], \tag{27}$$

$$\Sigma^{\theta'\theta}(p) = -\frac{S_d}{(2\pi)^d}\frac{g\nu}{2\varepsilon}[p^2 B_1 + (\mathbf{n}\cdot\mathbf{p})^2 B_2], \tag{28}$$

where (in weak small-scale anisotropy limit)

$$A_1 = \frac{1}{4d(d+2)(d+4)}[d(d-1)(d+4) + \alpha_1(d+1)(d+2) + \alpha_2(d-2) - 2\chi_1(d^2+3d+3) - \chi_2(3d-2) - 3\chi_3(3d+2)/(d+6)], \tag{29}$$

$$A_2 = \frac{1}{2d(d+2)(d+4)}\{-\alpha_1(d-2) + \alpha_2(d^3+2d^2-6d-4)/2 - \chi_1(2-3d) - \chi_2(3d^3+5d^2-16d-8)/4 - \chi_3 d(3d^2+7d-2)/[4(d+6)]\}, \tag{30}$$

$$A_3 = \frac{1}{d(d+2)(d+4)}\{(\alpha_2-\alpha_1)(d+1) + \chi_1(2d+1) - \chi_2(d^3+d^2+4d+8)/8 - \chi_3 d(d^2-d+22)/[8(d+6)]\}, \tag{31}$$

$$A_4 = -\frac{\chi_3(d-10)}{2(d+2)(d+6)}, \tag{32}$$

and

$$B_1 = \frac{1}{2d(d+2)(u+1)}\{(d-1)(d+2) + \alpha_1(d+1) + \alpha_2 - [\chi_1(d+1)+\chi_2](u+2)/(u+1) - 3\chi_3(u+2)/[(d+4)(u+1)] - \tau u(d+1)/(u+1)\}, \tag{33}$$

$$B_2 = \frac{1}{2d(d+2)(u+1)}[-2\alpha_1 + \alpha_2(d^2-2) + 2\chi_1(u+2)/(u+1) - \chi_2(d^2-2)(u+2)/(u+1) - \chi_3(d-2)(d+2)(u+2)/[(d+4)(u+1)] + 2\tau u/(u+1)]. \tag{34}$$

In Eqs. (27) and (28), $S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the area of the surface of the d-dimensional unit sphere. Thus, the renormalization constants Z_i ($i=1,\ldots,6$) are given as follows:

$$Z_1 = 1 - \frac{\bar{g}}{2\varepsilon}A_1, \quad Z_{j+1} = 1 - \frac{\bar{g}}{2\varepsilon}\frac{A_{j+1}}{\chi_j}, \tag{35}$$

$$Z_5 = 1 - \frac{\bar{g}}{2\varepsilon}\frac{B_1}{u}, \quad Z_6 = 1 - \frac{\bar{g}}{2\varepsilon}\frac{B_2}{u\tau}, \tag{36}$$

where we have also introduced suitable notation $\bar{g} = gS_d/(2\pi)^d$ and $j=1,2,3$.

In what follows, we shall be interested in the behavior of the equal-time structure functions of the scalar field θ, namely,

$$S_N(r) \equiv \langle [\theta(t,\mathbf{x}) - \theta(t,\mathbf{x}')]^N \rangle, \quad r = |\mathbf{x}-\mathbf{x}'| \tag{37}$$

in the inertial range specified by the inequalities $l = 1/\Lambda \ll r \ll L = 1/m$ (l is an internal length). In the field theoretic formulation of our stochastic problem the angular brackets $\langle\cdots\rangle$ mean the functional average over fields $\theta, \theta', \mathbf{v}', \mathbf{v}$ with weight $\exp(S_R)$ and independence of the original unrenormalized model of the scale-setting parameter μ of the renormalized model yields the RG differential equations for the renormalized structure functions (in general for arbitrary correlation functions) of the scalar field

$$\left[\mathcal{D}_\mu + \sum_{i=g,u,\chi_j,\tau}\beta_i\partial_i - \gamma_\nu\mathcal{D}_\nu\right]S_N^R = 0. \tag{38}$$

Here, $\mathcal{D}_x \equiv x\partial_x$ stands for any variable x and the RG functions (the β and the γ functions) are given by the well-known definitions [20,21]. In our case, using relations (24) for the renormalization constants, they acquire the following form:

$$\gamma_i \equiv \mathcal{D}_\mu \ln Z_i \tag{39}$$

for any renormalization constant Z_i ($i=1,\ldots,6$), and

$$\beta_g \equiv \mathcal{D}_\mu g = g(-2\varepsilon + 3\gamma_1), \tag{40}$$

$$\beta_u \equiv \mathcal{D}_\mu u = u(\gamma_1 - \gamma_5), \tag{41}$$

$$\beta_{\chi_j} \equiv \mathcal{D}_\mu \chi_j = \chi_j(\gamma_1 - \gamma_{j+1}), \quad j=1,2,3, \tag{42}$$

$$\beta_\tau \equiv \mathcal{D}_\mu \tau = \tau(\gamma_5 - \gamma_6). \tag{43}$$

Now, using the definition of the anomalous dimensions γ_i ($i=1,\ldots,6$) in Eq. (39) together with the renormalization constants given in Eqs. (35) and (36), one comes to the following expressions:

$$\gamma_1 = \bar{g}A_1, \quad \gamma_{j+1} = \frac{\bar{g}A_{j+1}}{\chi_j}, \quad j=1,2,3, \tag{44}$$

$$\gamma_5 = \frac{\bar{g}B_1}{u}, \quad \gamma_6 = \frac{\bar{g}B_2}{u\tau}. \tag{45}$$

In the next section we shall use these results for the investigation of stability of the nontrivial scaling regime of the model.

IV. STABILITY OF THE SCALING REGIME

Possible scaling regimes of a renormalized model are defined by the IR stable fixed points of the corresponding system of the RG equations [20,21]. The fixed point of the RG equations is defined by β functions, namely, by requirement of their vanishing. In our case we have the system of six β functions as given in Eqs. (40)–(43). Thus, the coordinates of the fixed point are determined by the system of six equations,

$$\beta_C(C_*) = 0, \tag{46}$$

where we have denoted $C=\{g,u,\chi_i,\tau\}$ ($i=1,2,3$) and C_* represents the corresponding value at the fixed point. The IR stability of the fixed point is given by the positive real parts of the eigenvalues of the matrix

$$\omega_{lm} = \left(\frac{\partial \beta_{C_l}}{\partial C_m}\right)_{C=C_*}, \quad l,m=1,\ldots,6. \tag{47}$$

In the weak small-scale anisotropy case we suppose that the coordinates of the fixed point are linear functions of the anisotropy parameters of the model, namely, α_1 and α_2. Using this assumption the coordinates of the fixed point can be found analytically in the following form:

$$g_* = g_{0*} + \alpha_1 g_{1*} + \alpha_2 g_{2*}, \tag{48}$$

$$u_* = u_{0*} + \alpha_1 u_{1*} + \alpha_2 u_{2*}, \tag{49}$$

$$\chi_{1*} = \alpha_1 \chi_{11*} + \alpha_2 \chi_{12*}, \quad \chi_{2*} = \alpha_1 \chi_{21*} + \alpha_2 \chi_{22*}, \tag{50}$$

$$\chi_{3*} = \alpha_1 \chi_{31*} + \alpha_2 \chi_{32*}, \quad \tau_* = \alpha_1 \tau_{1*} + \alpha_2 \tau_{2*}, \tag{51}$$

where explicit expressions for coefficients $g_{j*}, u_{j*}, \chi_{ik*}, \tau_{k*}$ for $i=1,2,3$; $j=0,1,2$; and $k=1,2$ are given in Appendix A.

To investigate the stability of the fixed point, it is necessary to apply it in the matrix of the first derivatives given in Eq. (47). In our case the matrix can be written in the following form:

$$\omega_{ij} = \begin{pmatrix} 2\varepsilon & \frac{\partial\beta_g}{\partial\chi_1} & \frac{\partial\beta_g}{\partial\chi_2} & \frac{\partial\beta_g}{\partial\chi_3} & 0 & 0 \\ 0 & \frac{\partial\beta_{\chi_1}}{\partial\chi_1} & \frac{\partial\beta_{\chi_1}}{\partial\chi_2} & \frac{\partial\beta_{\chi_1}}{\partial\chi_3} & 0 & 0 \\ 0 & \frac{\partial\beta_{\chi_2}}{\partial\chi_1} & \frac{\partial\beta_{\chi_2}}{\partial\chi_2} & \frac{\partial\beta_{\chi_2}}{\partial\chi_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial\beta_{\chi_3}}{\partial\chi_3} & 0 & 0 \\ 0 & \frac{\partial\beta_u}{\partial\chi_1} & \frac{\partial\beta_u}{\partial\chi_2} & \frac{\partial\beta_u}{\partial\chi_3} & \frac{\partial\beta_u}{\partial u} & \frac{\partial\beta_u}{\partial\tau} \\ 0 & \frac{\partial\beta_\tau}{\partial\chi_1} & \frac{\partial\beta_\tau}{\partial\chi_2} & \frac{\partial\beta_\tau}{\partial\chi_3} & \frac{\partial\beta_\tau}{\partial u} & \frac{\partial\beta_\tau}{\partial\tau} \end{pmatrix}_{C=C_*}. \tag{52}$$

Therefore, the eigenvalues of the matrix are

$$\lambda_1 = 2\varepsilon, \tag{53}$$

$$\lambda_{2,3} = \frac{1}{2}\left(\frac{\partial\beta_{\chi_1}}{\partial\chi_1} + \frac{\partial\beta_{\chi_2}}{\partial\chi_2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{\partial\beta_{\chi_1}}{\partial\chi_1} - \frac{\partial\beta_{\chi_2}}{\partial\chi_2}\right)^2 + 4\frac{\partial\beta_{\chi_1}}{\partial\chi_2}\frac{\partial\beta_{\chi_2}}{\partial\chi_1}}, \tag{54}$$

$$\lambda_4 = \frac{\partial\beta_{\chi_3}}{\partial\chi_3}, \tag{55}$$

$$\lambda_{5,6} = \frac{1}{2}\left(\frac{\partial\beta_u}{\partial u} + \frac{\partial\beta_\tau}{\partial\tau}\right) \pm \frac{1}{2}\sqrt{\left(\frac{\partial\beta_u}{\partial u} - \frac{\partial\beta_\tau}{\partial\tau}\right)^2 + 4\frac{\partial\beta_u}{\partial\tau}\frac{\partial\beta_\tau}{\partial u}}, \tag{56}$$

where all quantities are taken at the fixed point.

The eigenvalues λ_2, λ_3, λ_5, and λ_6 are rather complicated functions and we shall not present their explicit form here. On the other hand, it can be shown numerically that for $d > 2$, $\varepsilon > 0$, and for small enough absolute values of anisotropy parameters α_1 and α_2 (compatible with the general assumption of weak anisotropy) they are positive; thus, they do not influence the stability of the scaling regime. The opposite situation is given by eigenvalue λ_4. In the weak anisotropy case its explicit form reads

$$\lambda_4 = \frac{2\varepsilon(c_0 + \alpha_1 c_1 + \alpha_2 c_2)}{3d(d-1)(d+6)(3d^4 + 4d^3 - 5d^2 + 10d - 56)}, \tag{57}$$

with

$$c_0 = d(1456 - 652d + 144d^2 - 129d^3 - 55d^4 + 25d^5 + 3d^6),$$

$$c_1 = -2(320 + 48d - 18d^2 - 39d^3 - 26d^4 + 3d^5),$$

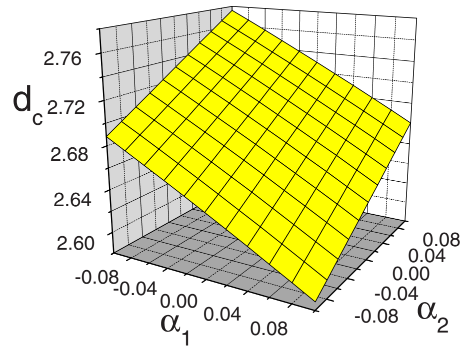

FIG. 4. (Color online) Dependence of the borderline dimension d_c on the anisotropy parameters α_1 and α_2 in the weak anisotropy limit.

$$c_2 = 2(160 + 504d - 102d^2 + 15d^3 - 61d^4 + 6d^5),$$

and it has the solution $\lambda_4=0$ within the interval of spatial dimensions $d \in (2,3]$. In the isotropic limit, i.e., for $\alpha_1 \to 0$ and $\alpha_2 \to 0$, this solution obtains simple form [25]

$$d_c = \frac{3\sqrt{17}-7}{2} \approx 2.684\,66, \tag{58}$$

where the so-called borderline dimension is denoted as d_c (for $d > d_c$ the Kolmogorov scaling regime is stable and for $d < d_c$ it is unstable, i.e., there is no scaling behavior). On the other hand, in the small-scale anisotropy case, borderline dimension d_c becomes a function of anisotropy parameters α_1 and α_2. The corresponding dependence in weak anisotropy case (for relatively small absolute values of the anisotropy parameters) is shown in Fig. 4. The most important conclusion for our further analysis is the fact that the weak anisotropy does not disturb the stability of the three-dimensional system, i.e., the borderline dimension d_c as a function of the anisotropy parameters is less than 3 for small enough anisotropy parameters.

However, it must be mentioned that the RG analysis briefly discussed above gives the same results as the analysis present in Ref. [54] (see also Ref. [55], where the same method is used in the analysis of the weak anisotropy limit of magnetohydrodynamic (MHD) turbulence). In Ref. [54] the correct treatment of the problem of stability of the scaling regime of the present model, at the first time studied in Ref. [56], was given (see also Ref. [57], where the problem of anisotropy in the stochastic Navier-Stokes equation was studied).

In general, the issue of interest is especially multiplicatively renormalizable equal-time two-point quantities $G(r)$ (see also, e.g., Ref. [30]). Examples of such quantities are the equal-time structure functions in the inertial interval as they were defined in Eq. (37). The IR scaling behavior of the function $G(r)$ (for $r/l \gg 1$ and any fixed r/L)

$$G(r) \simeq \nu_0^{d_G^\omega} l^{-d_G} (r/l)^{-\Delta_G} R(r/L) \tag{59}$$

is related to the existence of the IR stable fixed point of the RG equations (see above). In Eq. (59), d_G^ω and d_G are the corresponding canonical dimensions of the function G (the canonical dimensions of the model are given in Sec. III); $R(r/L)$ is the so-called scaling function, which cannot be determined by the RG equation (see, e.g., Ref. [21]); and Δ_G is the critical dimension defined as

$$\Delta_G = d_G^k + \Delta_\omega d_G^\omega + \gamma_G^*. \tag{60}$$

Here, γ_G^* is the fixed point value of the anomalous dimension $\gamma_G \equiv \mu\partial_\mu \ln Z_G$, where Z_G is the renormalization constant of the multiplicatively renormalizable quantity G, i.e., $G = Z_G G^R$ [31], and $\Delta_\omega = 2 - \gamma_\nu^*$ is the critical dimension of the frequency with $\gamma_\nu^* = \gamma_1^*$, which is defined in Eq. (44) and γ_1^* means that γ_1 is taken at the fixed point. From Eq. (40) one finds $\gamma_\nu^* \equiv \gamma_1^* = 2\varepsilon/3$. It is exact one-loop result, i.e., no higher-loop corrections exist. It means that the critical dimension of frequency is also known exactly, namely, $\Delta_\omega = 2(1-\varepsilon/3)$, as well as the critical dimensions of the fields,

$$\Delta_{\mathbf{v}} = 1 - \frac{2\varepsilon}{3}, \quad \Delta_{\mathbf{v}'} = d - 1 + \frac{2\varepsilon}{3}, \tag{61}$$

$$\Delta_\theta = -1 + \frac{\varepsilon}{3}, \quad \Delta_{\theta'} = d + 1 - \frac{\varepsilon}{3}. \tag{62}$$

The renormalized function G^R must satisfy the RG equation of the form

$$(\mathcal{D}_{RG} + \gamma_G) G^R(r) = 0, \tag{63}$$

with operator $\mathcal{D}_{RG}$ given explicitly in Eq. (38), namely,

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \sum_{i=g,u,\chi_j,\tau} \beta_i \partial_i - \gamma_\nu \mathcal{D}_\nu. \tag{64}$$

The difference between the functions G and G^R is only in the normalization, choice of parameters (bare or renormalized), and related to this choice the form of the perturbation theory (in g_0 or in g). The existence of a nontrivial IR stable fixed point means that in the IR asymptotic region $r/l \gg 1$ and any fixed r/L the function $G(r)$ takes on the self-similar form given in Eq. (59). As was already mentioned the scaling function $R(r/L)$ is not determined by the RG equation itself. The dependence of the scaling functions on the argument r/L in the region $r/L \ll 1$ can be studied using the well-known Wilson operator-product expansion (OPE) [20,21,24,25]. It shows that, in the limit $r/L \to 0$, the function $R(r/L)$ can be written in the following asymptotic form:

$$R(r/L) = \sum_i C_{F_i}(r/L)(r/L)^{\Delta_{F_i}}, \tag{65}$$

where C_{F_i} are coefficients regular in r/L. In general, the summation is implied over certain renormalized composite operators F_i with critical dimensions Δ_{F_i}. In the case under consideration the leading contribution is given by operators F_i having the form $F[N,p] = \partial_{i_1}\theta \cdots \partial_{i_p}\theta(\partial_i\theta\partial_i\theta)^n$ with $N = p + 2n$. In Sec. VI, we shall consider them in detail, where the complete one-loop calculation of the critical dimensions of the composite operators $F_{[N,p]}$ will be given for arbitrary values of N, d, and small absolute values of anisotropy parameters $\alpha_{1,2}$.

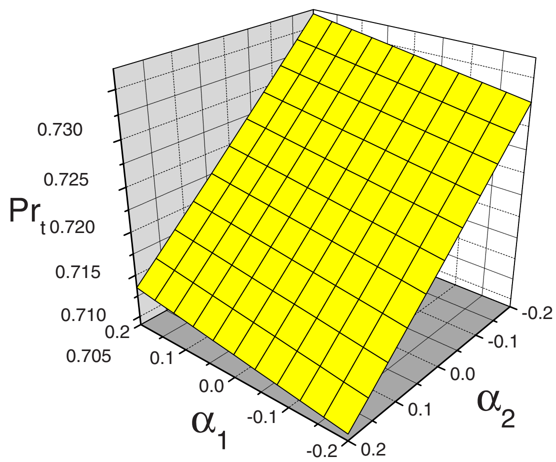

FIG. 5. (Color online) Dependence of the Prandtl number $\mathrm{Pr_t}$ on the anisotropy parameters α_1 and α_2 in the weak anisotropy limit.

V. ANISOTROPIC PRANDTL NUMBER

Before we shall investigate the anomalous scaling of the structure functions of the advected scalar field, let us briefly discuss the dependence of the so-called turbulent Prandtl number on the anisotropy parameters. The molecular Prandtl number Pr is defined as the dimensionless ratio of the coefficient of kinetic viscosity ν_0 and the coefficient of diffusivity $\nu_0 u_0$ [see Eqs. (1) and (2)]. In the case of fully developed turbulence, diffusion processes are radically accelerated and the concept of the universal turbulent Prandtl number $\mathrm{Pr_t}$ emerges as the ratio of the corresponding coefficient of turbulent viscosity and the coefficient of turbulent diffusivity. Here, the universality of the turbulent Prandtl number means its independence of individual properties of the fluid. Recently, the problem of the turbulent Prandtl number in the isotropic case was studied by the field theoretic approach in Ref. [58], where two-loop corrections to this quantity were calculated. It was shown that the turbulent Prandtl number is perturbatively rather stable, namely, the two-loop contributions are less than 10% of the one-loop result (see Ref. [58] for details) and the obtained result is in very good agreement with experimental values [1,59,60].

On the other hand, the presence of the small-scale anisotropy in the system under consideration leads to the dependence of the turbulent Prandtl number on the anisotropy parameters. It can be shown [58] that within the one-loop approximation $\mathrm{Pr_t}$ is given directly by the fixed point value of the parameter u, namely, $\mathrm{Pr_t}=1/u_*$. Therefore, applying it in our case, the dependence of the turbulent Prandtl number on the parameters of anisotropy in the weak anisotropy limit in one-loop approximation is given by the following simple relation:

$$\mathrm{Pr_t} = \frac{1}{u_{0*}+\alpha_1 u_{1*}+\alpha_2 u_{2*}}, \tag{66}$$

where explicit form of the coefficients u_{j*} for $j=0,1,2$ are given in Appendix A. In Fig. 5, the dependence can be seen explicitly. It can be seen that the one-loop weak anisotropy corrections to the one-loop isotropic turbulent Prandtl number, $\mathrm{Pr_t^{iso}}=0.7179$ [58], are smaller than the two-loop isotropic contribution and are in the region of $\pm 2\%$ of the isotropic value. Nevertheless, the situation can be considerably different when no restrictions on the uniaxial anisotropy parameters will be assumed. But this question is out of scope of the present paper and will be analyzed elsewhere.

VI. CRITICAL DIMENSIONS OF COMPOSITE OPERATORS AND ANOMALOUS SCALING

A. Operator product expansion

Using the OPE [20,21,24,25], the equal-time product $F_1(x')F_2(x'')$ of two renormalized composite operators [61] at $\mathbf{x}=(\mathbf{x}'+\mathbf{x}'')/2=\text{const}$ and $\mathbf{r}=\mathbf{x}'-\mathbf{x}''\to 0$ can be written in the following form:

$$F_1(x')F_2(x'') = \sum_i C_{F_i}(\mathbf{r})F_i(\mathbf{x},t), \tag{67}$$

where the summation is taken over all possible renormalized local composite operators F_i allowed by symmetry with definite critical dimensions Δ_{F_i}, and functions C_{F_i} are the corresponding Wilson coefficients. The renormalized correlation function $\langle F_1(x')F_2(x'')\rangle$ can now be found by averaging Eq. (67) with the weight $\exp S^R$ with S^R from Eq. (23). The quantities $\langle F_i\rangle$ appear on the right-hand side and their asymptotic behavior in the limit $L^{-1}\to 0$ is then found from the corresponding RG equations and has the form $\langle F_i\rangle \propto L^{-\Delta_{F_i}}$. From the OPE (67) one can find that the scaling function $R(r/L)$ in representation (59) for the correlation function $F_1(x')F_2(x'')$ has the form given in Eq. (65), where the coefficients C_{F_i} are regular in $(r/L)^2$.

The specific feature of the turbulence models is the existence of operators with negative critical dimensions (the so-called "dangerous" operators) [21,22,24,25,27]. Their presence in the OPE determines the IR behavior of the scaling functions and leads to their singular dependence on L when $r/L\to 0$. If the spectrum of the dimensions Δ_{F_i} for a given scaling function is bounded from below, the leading term of its behavior for $r/L\to 0$ is given by the minimal dimension. At this point the turbulence models are crucially different from the models of critical phenomena, where the leading contribution to representation (59) is given by the simplest operator $F=1$ with the dimension $\Delta_F=0$, and the other operators determine only the corrections that vanish for $r/L\to 0$.

In what follows, our aim is to investigate the behavior of the equal-time structure functions of the scalar field as defined in Eq. (37). In this case, representation (59) is valid with the following dimensions:

$$d_G^{\omega} = -\frac{N}{2}, \quad d_G = -N, \quad \Delta_G = N\left(-1+\frac{\varepsilon}{3}\right). \tag{68}$$

In general, not only do the operators which are present in the corresponding Taylor expansion are entering into the OPE but also all possible operators that admix to them in renormalization. In our anisotropic model the leading contribution of the Taylor expansion for the structure functions (37) is given by the tensor composite operators constructed solely of the scalar gradients

$$F[N,p] \equiv \partial_{i_1}\theta \cdots \partial_{i_p}\theta(\partial_i\theta\partial_i\theta)^n, \tag{69}$$

where $N=p+2n$ is the total number of the fields θ entering into the operator and p is the number of the free vector indices (see, e.g., Ref. [26] for details).

B. Composite operators $F[N,p]$: Renormalization and critical dimensions

The necessity of additional renormalization of the composite operators (69) is related to the fact that the coincidence of the field arguments in Green's functions containing them leads to additional UV divergences. These divergences must be removed by special kind of renormalization procedure, which can be found, e.g., in Refs. [19–21,24,25]. Besides, typically, the composite operators are mixed under renormalization.

Thus, let $F \equiv \{F_\alpha\}$ be a closed set of composite operators which are mixed only with each other in renormalization. Then the renormalization matrix $Z_F \equiv \{Z_{\alpha\beta}\}$ and the matrix of corresponding anomalous dimensions $\gamma_F \equiv \{\gamma_{\alpha\beta}\}$ for this set are given as follows:

$$F_\alpha = \sum_\beta Z_{\alpha\beta} F_\beta^R, \quad \gamma_F = Z_F^{-1}\widetilde{D}_\mu Z_F. \tag{70}$$

Renormalized composite operators are subjected to the following RG differential equations:

$$\left(\mathcal{D}_\mu + \sum_{i=g,u,\chi_j,\tau} \beta_i\partial_i - \gamma_\nu\mathcal{D}_\nu\right)F_\alpha^R = -\sum_\beta \gamma_{\alpha\beta}F_\beta^R, \tag{71}$$

which lead to the following matrix of critical dimensions $\Delta_F \equiv \{\Delta_{\alpha\beta}\}$:

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \quad \Delta_\omega = 2 - \frac{2\varepsilon}{3}, \tag{72}$$

where d_F^k and d_F^ω are diagonal matrices of the corresponding canonical dimensions and γ_F^* is the matrix of anomalous dimensions (70) taken at the fixed point. In the end, the critical dimensions of the set of operators $F \equiv \{F_\alpha\}$ are given by the eigenvalues of the matrix Δ_F. The so-called "basis" operators that possess definite critical dimensions have the form

$$F_\alpha^{bas} = \sum_\beta U_{\alpha\beta}F_\beta^R, \tag{73}$$

where the matrix $U_F = \{U_{\alpha\beta}\}$ is such that $\Delta_F' = U_F\Delta_F U_F^{-1}$ is diagonal.

As was already mentioned, in what follows, the central role is played by tensor composite operators $\partial_{i_1}\theta \cdots \partial_{i_p}\theta(\partial_i\theta\partial_i\theta)^n$, constructed solely of the scalar gradients. For further convenience it is useful to deal with the scalar operators obtained by contracting the tensors with the appropriate number of the uniaxial anisotropy vectors $\mathbf{n}$ [26],

$$F[N,p] \equiv [(\mathbf{n}\cdot\partial)\theta]^p(\partial_i\theta\partial_i\theta)^n, \quad N \equiv 2n+p. \tag{74}$$

Detailed analysis shows that the composite operators (74) with different N are not mixed in renormalization, and therefore the corresponding renormalization matrix $Z_{[N,p][N',p']}$ is in fact block diagonal, i.e., $Z_{[N,p][N',p']}=0$ for $N' \neq N$ [26].

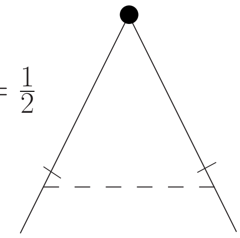

FIG. 6. Graphical representation of the one-loop correction to Γ_N in Eq. (77).

Further reduction in the matrix elements of the matrix of renormalization constants Z is obtained in the isotropic case, as well as in the case when the large-scale anisotropy is present. In these situations the elements $Z_{[N,p][N,p']}$ vanish for $p<p'$; thus, the block $Z_{[N,p][N,p']}$ is in fact triangular along with the corresponding blocks of the matrices U_F and Δ_F from Eqs. (73) and (72). In the isotropic case it can be diagonalized by changing to irreducible operators (scalars, vectors, and traceless tensors), but even for nonzero imposed gradient its eigenvalues are the same as in the isotropic case. Therefore, the inclusion of large-scale anisotropy does not affect critical dimensions of the operators (74). On the other hand, in the case when small-scale anisotropy is present, the operators with different values of p mix heavily in renormalization, and the matrix $Z_{[N,p][N,p']}$ is neither diagonal nor triangular here and one can write

$$F[N,p] = \sum_{l=0}^{\lfloor N/2 \rfloor} Z_{[N,p][N,N-2l]}F^R[N,N-2l], \tag{75}$$

where $\lfloor N/2 \rfloor$ means the integer part of $N/2$. Therefore, each block of renormalization constants with a given N is an $(\lfloor N/2\rfloor+1)\times(\lfloor N/2\rfloor+1)$ matrix. Of course, the matrix of critical dimensions (72), whose eigenvalues at the IR stable fixed point are the critical dimensions $\Delta[N,p]$ of the set of operators $F[N,p]$, has also a dimension $(\lfloor N/2\rfloor+1)\times(\lfloor N/2\rfloor+1)$.

To obtain the renormalization constants $Z_{[N,p][N,p']}$ we are interested in the Nth term of the expansion of the generating functional $\Gamma(x;\theta)$ of the one-irreducible Green's functions with one composite operator $F[N,p]$ from Eq. (74) and any number of fields θ. We denote it as $\Gamma_N(x;\theta)$ and it has the following form:

$$\Gamma_N(x;\theta) = \frac{1}{N!}\int dx_1 \cdots \int dx_N\ \theta(x_1)\cdots\theta(x_N)\langle F[N,p] \times(x)\theta(x_1)\cdots\theta(x_N)\rangle_{1\text{-ir}}. \tag{76}$$

In the one-loop approximation it is given as

$$\Gamma_N = F[N,p] + \Gamma^{(1)}, \tag{77}$$

where $\Gamma^{(1)}$ is given by the analytical calculation of the diagram in Fig. 6, and the first term in Eq. (77) represents "tree" approximation (see also Ref. [26]).

The black circle with two attached lines in the diagram in Fig. 6 denotes the variational derivative $V(x;x_1,x_2) \equiv \delta^2F[N,p]/\delta\theta(x_1)\delta\theta(x_2)$, where the second variation makes needed combinatorics, namely, the operator $F[N,p]$ contains

N fields θ and one must take two of them (in all possible ways) to construct the one-loop diagram as shown in Fig. 6. It can be represented in the following convenient form [26]:

$$V(x;x_1,x_2) = \partial_i \delta(x-x_1)\partial_j \delta(x-x_2)\frac{\partial^2}{\partial a_i \, \partial a_j}[(\mathbf{n}\cdot\mathbf{a})^p(a^2)^n], \tag{78}$$

where a constant vector a_i will be substituted with $\partial_i\theta(x)$ after the differentiation (see below). The analytical form of the diagram in Fig. 6 (without the symmetry factor 1/2) is the following:

$$\int dx_1 \cdots \int dx_4 \; V(x;x_1,x_2)\langle\theta(x_1)\theta'(x_3)\rangle_0\langle\theta(x_2)\theta'(x_4)\rangle_0 \times\langle v_k(x_3)v_l(x_4)\rangle_0\partial_k\theta(x_3)\partial_l\theta(x_4), \tag{79}$$

where the bare propagators are given in Eqs. (10) and (12) and the derivatives are related to the ordinary vertex factors shown in the right figure in Fig. 2.

We are interested in the UV divergent part of expression (79), which is needed for the determination of the corresponding renormalization constants. But the needed UV divergent part is proportional to the polynomial built of N gradients $\partial_i\theta(x)$ at a single space-time point x, and all of them have been already extracted from Eq. (79), namely, $N-2$ gradients are given by vertex (78) and the other two gradients are given by the ordinary vertex factors in Fig. 1. This important point allows us to replace the gradients with the constant vectors $\mathbf{a}$. Therefore, the divergent part of expression (79) can be written in the following compact form:

$$a_k a_l \frac{\partial^2}{\partial a_i \, \partial a_j}[(\mathbf{n}\cdot\mathbf{a})^p(a^2)^n]X_{ij,kl}, \tag{80}$$

with

$$X_{ij,kl} \equiv \int dx_3 \int dx_4 \partial_i\langle\theta(x)\theta'(x_3)\rangle_0\partial_j\langle\theta(x)\theta'(x_4)\rangle_0 \times\langle v_k(x_3)v_l(x_4)\rangle_0. \tag{81}$$

After rather time consuming but straightforward calculations one obtains the following result for the quantity defined in Eq. (80):

$$\frac{S_d}{(2\pi)^d}\frac{g}{8u(1+u)^2}\frac{1}{d(d+2)(d+4)}\left(\frac{\mu}{m}\right)^{2\varepsilon}\frac{1}{\varepsilon} \times\{Q_1 F[N,p-2]+Q_2 F[N,p]+Q_3 F[N,p+2]\}, \tag{82}$$

where we have substituted the unrenormalized quantities with the renormalized one, a_i have been replaced with the gradients $\partial_i\theta(x)$ (thus, they again form the operators $F[N,q]$, with $q=p-2,p,p+2$), and the explicit form of the coefficients Q_i ($i=1,2,3$) is given in Appendix B. The absence of a term proportional to the operator $F[N,p+4]$ in Eq. (82), in contrast to the corresponding expressions obtained in Refs. [26,34] is related to the weak anisotropy limit, which is considered in the present paper.

Now, using the standard renormalization procedure the renormalization constants $Z_{[N,p][N,p']}$ defined in Eq. (75) are found from the requirement that function (77) is UV finite (i.e., contains no poles in ε) when it is written in renormalized variables and with the replacement $F[N,p]\to F^R[N,p]$. In the end, from Eqs. (77) and (82) we have

$$Z_{[N,p][N,p+2(i-2)]} = \delta_{2i} + \frac{S_d}{(2\pi)^d}\frac{g}{A}\frac{1}{2\varepsilon}Q_i, \quad i=1,2,3, \tag{83}$$

where $A=8u(1+u)^2 d(d+2)(d+4)$. Using the definition of the matrix of anomalous dimensions $\gamma_{[N,p][N',p']}$ given in Eq. (70), we are coming to the following result:

$$\gamma_{[N,p][N,p+2(i-2)]} = -\frac{S_d}{(2\pi)^d}\frac{g}{A}Q_i, \quad i=1,2,3, \tag{84}$$

and the desired matrix of critical dimensions (72) has the form

$$\Delta_{[N,p][N,p']} = \frac{N\varepsilon}{3} + \gamma^*_{[N,p][N,p']}, \tag{85}$$

where the asterisk means that the quantities are taken at the fixed point. The nonzero one-loop contribution to the matrix of critical dimension (85) is represented by Eq. (84). It means that the matrix elements of the matrix $\gamma_{[N,p][N',p']}$ other than given in Eq. (84) are equal to zero. It can be seen immediately that the matrix of critical dimensions depends on the anisotropy parameters α_1 and α_2.

In the end, the critical dimensions $\Delta[N,p]$ are given by the eigenvalues of matrix (85). In the isotropic limit ($\alpha_1=\alpha_2=0$) one comes to the triangular matrix; therefore, its eigenvalues are given directly by the diagonal elements. Besides, in the one-loop level the isotropic result is also independent of the reciprocal value of the Prandtl number u_*, namely,

$$\Delta[N,p] = \frac{N\varepsilon}{3} + \frac{2p(p-1)-(d-1)(N-p)(d+N+p)}{3(d-1)(d+2)}\varepsilon. \tag{86}$$

In general, Eq. (86) represents the critical dimensions for the model with the presence of large-scale anisotropy and the isotropic critical dimensions are obtained when one takes even values for N and puts $p=0$. This result is the same (up to normalization) as the isotropic results (or results with large-scale anisotropy) obtained for the toy models of turbulent advection with a given Gaussian statistics of the velocity field (Kraichnan's rapid-change model, frozen velocity field model, and models with finite correlations in time of the velocity field), i.e., roughly speaking, within the one-loop approximation the anomalous behavior of the single-time correlation functions [the structure functions (37) are the examples] does not depend on the statistics of the velocity field. As we shall see in the next section the situation is different when the isotropy of the system is broken and the small-scale anisotropy of the energy pumping is assumed. In this case, the matrix of critical dimensions is not diagonal and the eigenvalues depend on anisotropy parameters in different ways for different models. But, first, let us discuss the

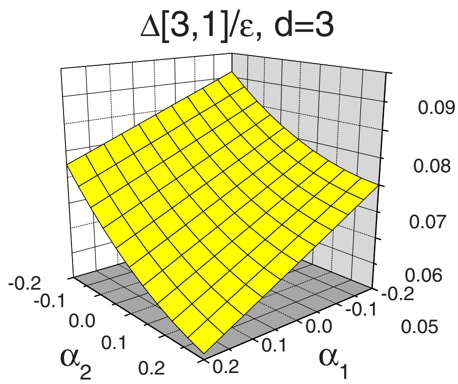

FIG. 7. (Color online) Dependence of the critical dimension $\Delta[3,1]/\varepsilon$ on the anisotropy parameters α_1 and α_2 in the weak anisotropy limit.

influence of the small-scale anisotropy on result (86) in the framework of the present model.

The fact that matrix (85) is triangular in the isotropic case (it is also triangular in the case with large-scale anisotropy) is important because it allows one to assign uniquely the concrete critical dimension to the corresponding composite operator even in the case with small-scale anisotropy and to study their hierarchical structures as functions of p (see Ref. [26] for details). As was shown in Ref. [26] within the Kraichnan model, as for anomalous scaling, the leading role is played by the operators with the most negative critical dimensions: for the structure functions (37) with even N, it is the operator with $p=0$ and for the structure functions (37) with odd N it is the operator with $p=1$. The same situation also holds in the present model with the velocity field driven by the Navier-Stokes velocity field.

In Figs. 7–11, the behavior of the minimal eigenvalues of the matrix of critical dimensions $\Delta[N,p]$ for various values of $N=3-7$ for even values of N and $p=1$ for odd values of N) is shown as a function of relatively small anisotropy parameters α_1 and α_2 in the weak anisotropy limit in the three-dimensional case. The dependence of the critical dimension $\Delta[2,0]$ is not shown explicitly because it must be identically equal to zero. It can be shown, for example, by using the so-called Schwinger equation (see, e.g., Ref. [26]).

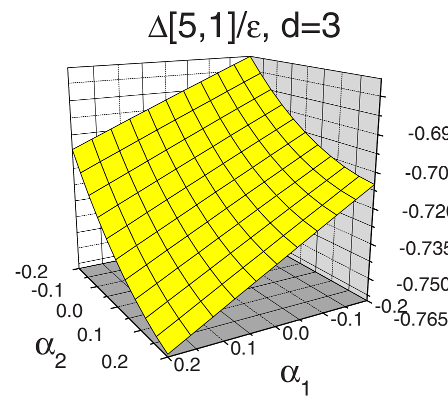

FIG. 9. (Color online) Dependence of the critical dimension $\Delta[5,1]/\varepsilon$ on the anisotropy parameters α_1 and α_2 in the weak anisotropy limit.

Dimensions $\Delta[N,p]$ obey the following important hierarchies:

$$\Delta[2n,0] > \Delta[2n+2,0], \tag{87}$$

$$\Delta[2n+1,1] > \Delta[2n+3,1], \tag{88}$$

$$\Delta[N,p] > \Delta[N,p'], \quad p > p'. \tag{89}$$

In the isotropic case ($\alpha_{1,2}=0$), their validity follows from Eq. (86). On the other hand, in the small-scale anisotropic case they follow from numerical investigations [see Figs. 5–11 to test relations (87) and (88)]. Relations (87)–(89) are important for the determination of asymptotic behavior of the single-time structure functions in the next section.

C. Anomalous scaling of the structure functions S_N

Now, we are ready to write down the final asymptotic expression for the structure functions (37). First of all, in the uniaxial anisotropy situation when the preferred direction is given by unit constant vector **n**, as defined in Eq. (8), the structure functions (37) can be decomposed in the following way (Legendre decomposition) [62]:

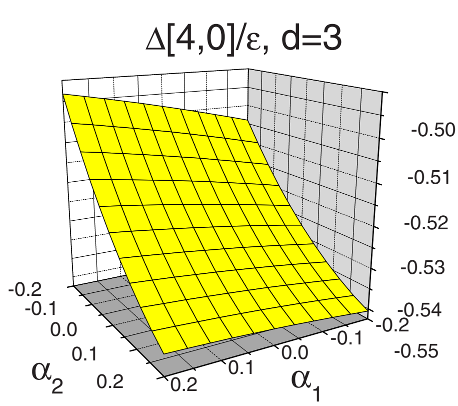

FIG. 8. (Color online) Dependence of the critical dimension $\Delta[4,0]/\varepsilon$ on the anisotropy parameters α_1 and α_2 in the weak anisotropy limit.

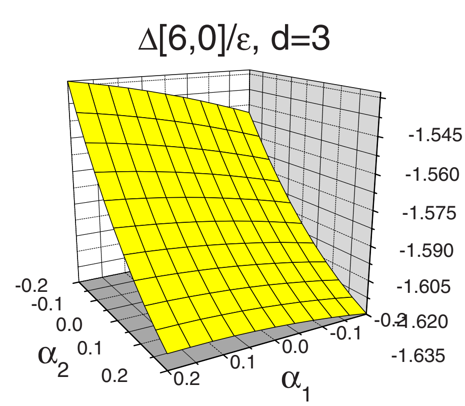

FIG. 10. (Color online) Dependence of the critical dimension $\Delta[6,0]/\varepsilon$ on the anisotropy parameters α_1 and α_2 in the weak anisotropy limit.

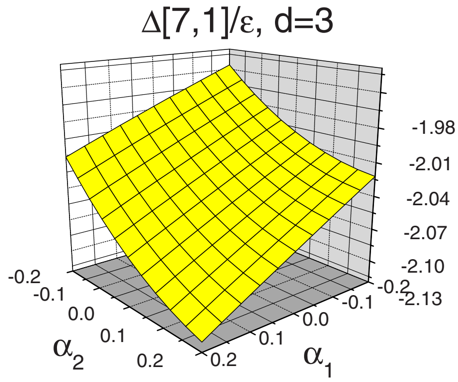

FIG. 11. (Color online) Dependence of the critical dimension $\Delta[7,1]/\varepsilon$ on the anisotropy parameters α_1 and α_2 in the weak anisotropy limit.

$$S_N(\mathbf{r}) = \sum_{p=0}^{\infty} S_{N,p}(r)P_p(z), \quad z = (\mathbf{n}\cdot\mathbf{r})/r, \tag{90}$$

where $P_p(z)$ are the so-called Gegenbauer polynomials (the d-dimensional generalization of the Legendre polynomials; see, e.g., Ref. [63]) and $S_{N,p}$ are the corresponding scalar coefficients that depend only on $r=|\mathbf{r}|$ in the case of large-scale anisotropy [30], but they can depend also on the anisotropy parameters in the case with small-scale anisotropy. Now, using the combination of the RG representation (59) for the decomposed S_N given in Eq. (90) with dimensions (68) together with the OPE (65) leads to the general asymptotic expression for the structure functions (37) within the inertial range, namely,

$$S_N(\mathbf{r}) \simeq r^{N(1-\varepsilon/3)} \sum_{N'\leq N} \sum_{p} \{C_{N',p}(r/L)^{\Delta[N',p]} + \cdots\}, \tag{91}$$

where p obtains all possible values for a given N', $C_{N',p}$ are numerical coefficients which are functions of the parameters of the model $(\varepsilon, d, \alpha_1, \alpha_2, z)$, the dimensions $\Delta[N',p]$ are given by the eigenvalues of the matrix of critical dimensions (85), and the Gegenbauer polynomials $P_p(z)$ from decomposition (90) are included in the coefficients $C_{N',p}$. In Eq. (91) the ellipsis means contributions by the operators others than $F[N,p]$ which are not important in the asymptotic regime (see, e.g., [21,26] for details). The appearance of dimensions $\Delta[N',p]$ with $p\neq 0$ on the right-hand side of Eq. (91) is related to the fact that, in the presence of small-scale anisotropy, the corresponding operators acquire nonzero mean value. On the other hand, the leading term for the small r/L behavior of the structure function S_N is given by a term with the minimal possible value of $\Delta[N',p]$. Using the relations given in Eqs. (87)–(89) the final asymptotic expressions for the single-time structure functions in the presence of weak uniaxial small-scale anisotropy are

$$S_N(\mathbf{r}) \simeq r^{N(1-\varepsilon/3)}(r/L)^{\Delta[N,0]} \tag{92}$$

for even value of N, and

$$S_N(\mathbf{r}) \simeq r^{N(1-\varepsilon/3)}(r/L)^{\Delta[N,1]} \tag{93}$$

for odd value of N with critical dimensions $\Delta[N,0]$ or $\Delta[N,1]$ given in Figs. 7–11.

In the end, let us briefly discuss the persistence of the anisotropy in the inertial range in the present model. It is well known that the persistence of anisotropy is given by the behavior of odd correlation functions and appropriate quantities for their investigations are the dimensionless ratios $R_{2k+1} \equiv S_{2k+1}/S_2^{k+1/2}$. Because $\Delta[2,0]=0$ then their explicit asymptotic form is ($N=2k+1$, $k\geq 1$)

$$R_N = \frac{S_N}{S_2^{N/2}} \simeq \left(\frac{r}{L}\right)^{\Delta[N,1]}, \tag{94}$$

and from the results shown in Figs. 7, 9, and 11 it is evident that the so-called skewness factor $R_3=S_3/S_2^{3/2}$ decreases for $r/L\to 0$ but much slower than it is expected from dimensional analysis, while the higher-order ratios R_{2k+1} ($k\geq 2$) increase. These results are qualitatively (and in the isotropic limit $\alpha_{1,2}\to 0$ also quantitatively) in agreement with those obtained in the framework of the investigation of the anomalous scaling of a passive quantity advected by the Gaussian velocity fields [5–9,17,26,30,34,41] (see also survey papers [18,62] and references cited therein).

VII. COMPARISON WITH THE ANISOTROPIC RAPID-CHANGE MODEL

As was already mentioned in Sec. I, in Refs. [26,34] the influence of uniaxial small-scale anisotropy on the anomalous scaling of the single-time structure functions of a passive scalar advected by the Gaussian velocity field with δ correlations in time [26] (the rapid-change model) and with finite time correlations [34] was studied. But these models can be considered only as toy models of real anisotropic turbulent advection; therefore, the natural question immediately arises, namely, what is the difference between the results obtained in those investigations and the results obtained in the framework of a more realistic nonlinear model considered in the present paper. The answer on this question is interesting at least because, on one hand, it can shown unambiguous difference between the results obtained within the model with a Gaussian statistics of velocity field (e.g., the rapid-change model) and the results obtained within the model with the non-Gaussian statistics of the velocity field (the velocity driven by the stochastic Navier-Stokes equation) even at the one-loop approximation level; and, on the other hand, it is also interesting for the determination of the importance of inclusion of nonlinearities of the velocity field (the existence of higher velocity correlations) for the investigation of anomalous scaling. For example, in Refs. [26,34] it was shown that in the framework of models with a Gaussian statistics of the velocity field there exist regions of anisotropy parameters (large enough positive values of α_1 and α_2) for which the critical dimension $\Delta[3,1]$ becomes negative and the skewness factor R_3 (see Eq. (94) for $N=3$) increases going down toward the depth of the inertial interval in contrast to the isotropic case and to the case when anisotropy parameters are not very large. The open question is whether such a behavior is also held in the model with the velocity field driven by the stochastic Navier-Stokes equation or it is only an artifact related to the Gaussian statistics of the velocity field. However, when one works with weak anisotropy limit (our case), the ultimate answer cannot be done. Thus, the question is still open and will be considered elsewhere. Thus, in what follows, we shall try to compare our results to those obtained in Ref. [26], namely, we shall try to compare the corresponding critical dimensions $\Delta[N,p]$ as functions of anisotropy parameters which drive the anomalous scaling of the structure functions of scalar fields in both models.

However, first of all, a short remark must be done. In the present model the small-scale anisotropy is introduced into the system by the geometric properties of the statistics of the random force which drives the stochastic Navier-Stokes equation for the velocity field [see Eqs. (4) and (8)]. On the other hand, the anisotropic properties of the turbulent environment in the rapid-change model [26] (brief description will be also done below) is given directly by the Gaussian statistics of the velocity field. Nevertheless, the comparison can be done because the corresponding isotropic critical dimensions are the same for both models (up to normalization, see below) and, at the same time, the critical dimensions are the continuous functions of anisotropy parameters in the corresponding anisotropic models. Another fact, which must be also taken into account in what follows, is the fact that in our model we have supposed that the uniaxial anisotropy is weak, i.e., the anisotropy parameters are close to zero ($|\alpha_{1,2}| \ll 1$) and, as a result, only the linear parts of all quantities as functions of anisotropy parameters were taken into account during calculations (see Secs. III–VIII). On the other hand, in Ref. [26] the calculations were done with no restrictions on the value of anisotropy parameters. It means that to have relevant comparison of both models it is necessary to carry out all calculations given in Ref. [26] in the limit of weak anisotropy and only then the comparison of the results of both models can be done. This is actually done in what follows.

Let us briefly describe the rapid-change model with small-scale uniaxial anisotropy. The advection of a passive scalar field $\theta(x)$ in the rapid-change model is described by the stochastic equation (for details, see Ref. [26])

$$\partial_t \theta + v_i \partial_i \theta = \kappa_0 \Delta \theta + f^{\theta}, \tag{95}$$

where κ_0 is the molecular diffusivity coefficient (for further notation, see Sec. II) and the velocity field $\mathbf{v}(x)$ obeys a Gaussian distribution with zero mean and correlator

$$\langle v_i(x) v_j(x') \rangle = g_0 \kappa_0 \delta(t-t') \int \frac{d^d\mathbf{k}}{(2\pi)^d} R_{ij}(\mathbf{k}) k^{-d-\epsilon} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \tag{96}$$

Here, g_0 plays the role of a coupling constant, the anisotropic transverse tensor $R_{ij}(\mathbf{k})$ is given in Eq. (8), and $0<\epsilon<2$ is a parameter with the real (Kolmogorov) value $\epsilon=4/3$.

The detailed field theoretic RG analysis of the stochastic model defined by Eqs. (95) and (96) was done in Ref. [26], where the IR stable scaling regime was found with the corresponding coordinates of the fixed point and also the analysis of the scaling functions of the single-time structure functions of the scalar field was done by using the OPE, where the main contribution is again given by the composite operators given in Eq. (69). We have repeated all the necessary calculations within the rapid-change model in the weak anisotropy limit and the final asymptotic inertial interval expression for the single-time structure functions (37) is obtained in the form (see Refs. [26,34] for details)

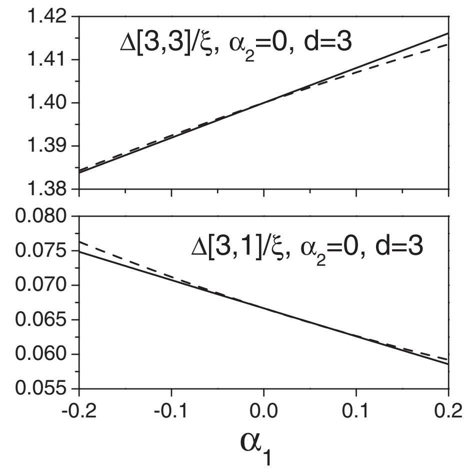

FIG. 12. Dependence of the critical dimension $\Delta[3,p]/\xi$ for $p=1,3$ on the anisotropy parameter α_1 for $\alpha_2=0$ in the weak anisotropy limit and for spatial dimension $d=3$. The solid line corresponds to the model of a passive scalar advection by the velocity field driven by the stochastic Navier-Stokes equation [$\xi=\varepsilon$, where ε is defined in Eq. (5)] and the dashed line corresponds to the rapid-change model [$\xi=3/2\epsilon$, where ϵ is defined in Eq. (96)].

$$S_N(\mathbf{r}) \simeq r^{N(1-\epsilon/2)} (r/L)^{\Delta[N,0]}, \tag{97}$$

for even value of N, and

$$S_N(\mathbf{r}) \simeq r^{N(1-\epsilon/2)} (r/L)^{\Delta[N,1]}, \tag{98}$$

for odd value of N, respectively, as a result of the fact that the hierarchy relations (87)–(89) hold. Here, $\Delta[N,p]$ are again the corresponding critical dimensions of the composite operators given in Eq. (69), but now they are calculated in the framework of the rapid-change model. In the isotropic case they are the same (up to normalization $\epsilon=2/3\varepsilon$) as those present in Eq. (86), namely,

$$\Delta[N,p] = \frac{N\epsilon}{2} + \frac{2p(p-1)-(d-1)(N-p)(d+N+p)}{2(d-1)(d+2)}\epsilon. \tag{99}$$

We shall not present here analytical expressions for the corresponding critical dimensions in weak anisotropy limit for the rapid-change model because they can be simply derived from expressions presented in Refs. [26,34]. Rather we shall concentrate on the numerical comparison of the corresponding results with those presented in the previous section in the framework of the model of a passive scalar advection by the corresponding Navier-Stokes velocity field. In Figs. 12–21 the behavior of the corresponding critical dimensions $\Delta[N,p]$ for $N=3,\dots,7$ and $p=0,2$ for even values of N and

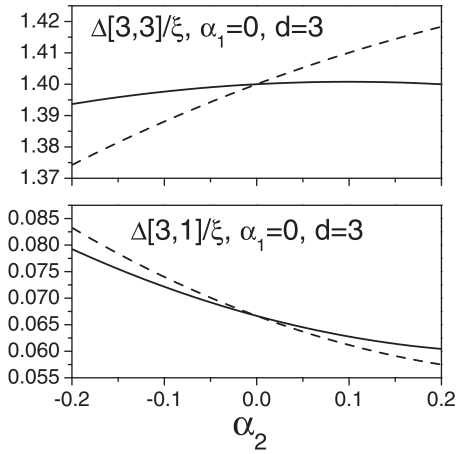

FIG. 13. Dependence of the critical dimension $\Delta[3,p]/\xi$ for $p=1,3$ on the anisotropy parameter α_2 for $\alpha_1=0$ in the weak anisotropy limit and for spatial dimension $d=3$ (for the notation, see the caption of Fig. 12).

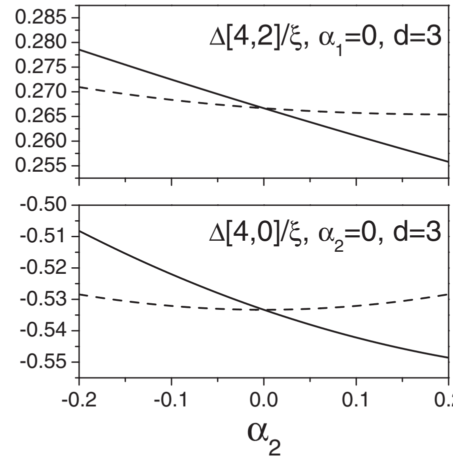

FIG. 15. Dependence of the critical dimension $\Delta[4,p]/\xi$ for $p=0,2$ on the anisotropy parameter α_2 for $\alpha_1=0$ in the weak anisotropy limit and for spatial dimension $d=3$ (for the notation, see the caption of Fig. 12).

$p=1,3$ for odd values of N is shown as functions of anisotropy parameters α_1 and α_2. It is shown that the critical dimensions which are related to the existence of anomalous scaling [see Eq. (94)] and which are universal (the same) for models with Gaussian and non-Gaussian statistics of the velocity field in the isotropic case (as well as in the case with large-scale anisotropy) at one-loop level of approximation are strongly nonuniversal in the case when the small-scale anisotropy is present.

VIII. CONCLUSION

Using the field theoretic RG technique and the operator product expansion, we have investigated the influence of weak uniaxial small-scale anisotropy on the behavior of the single-time structure functions of a passive scalar advected by the velocity field driven by the stochastic Navier-Stokes equation in one-loop approximation. First of all, the stability of the corresponding IR scaling regime for three-dimensional system as a function of anisotropy parameters was shown, which is given by the IR stable fixed point of the corresponding RG equations (see Fig. 4). We have also briefly investigated the influence of the small-scale anisotropy on the turbulent Prandtl number Pr_t. It was shown that in the weak anisotropy limit the dependence of Pr_t on the anisotropy parameters of the model is rather small (see Fig. 5), but the situation can be radically different when no restriction on the anisotropy parameters is assumed.

Further, we have studied the influence of small-scale uniaxial anisotropy on the anomalous scaling of the single-time structure functions of a passive scalar in the weak anisotropy limit by using the OPE. The leading composite operators with the smallest (the most negative) critical dimensions are studied in detail and the corresponding critical dimensions $\Delta[N,p]$ are found as functions of the anisotropy parameters. The persistence of anisotropy deep inside

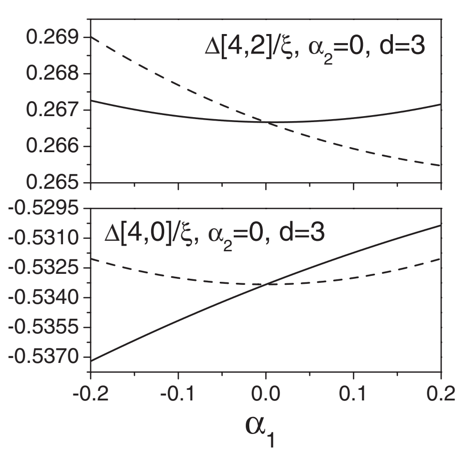

FIG. 14. Dependence of the critical dimension $\Delta[4,p]/\xi$ for $p=0,2$ on the anisotropy parameter α_1 for $\alpha_2=0$ in the weak anisotropy limit and for spatial dimension $d=3$ (for the notation, see the caption of Fig. 12).

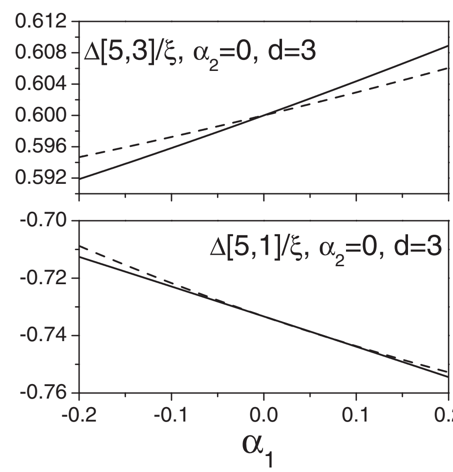

FIG. 16. Dependence of the critical dimension $\Delta[5,p]/\xi$ for $p=1,3$ on the anisotropy parameter α_1 for $\alpha_2=0$ in the weak anisotropy limit and for spatial dimension $d=3$ (for the notation, see the caption of Fig. 12).

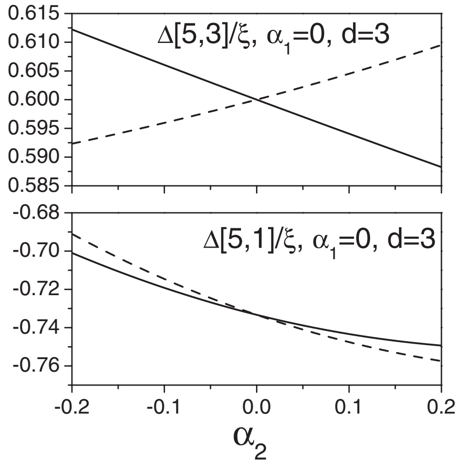

FIG. 17. Dependence of the critical dimension $\Delta[5,p]/\xi$ for $p=1,3$ on the anisotropy parameter α_2 for $\alpha_1=0$ in the weak anisotropy limit and for spatial dimension $d=3$ (for the notation, see the caption of Fig. 12).

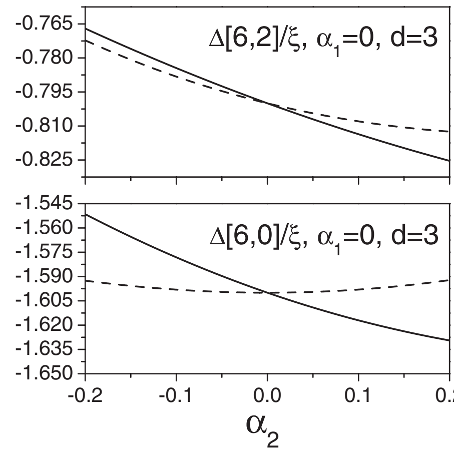

FIG. 19. Dependence of the critical dimension $\Delta[6,p]/\xi$ for $p=0,2$ on the anisotropy parameter α_2 for $\alpha_1=0$ in the weak anisotropy limit and for spatial dimension $d=3$ (for the notation, see the caption of Fig. 12).

the inertial range is demonstrated and briefly discussed by using the inertial range asymptotic behavior of the so-called skewness and hyperskewness factors [see Eq. (94)]. It is shown that the corresponding anomalous dimensions, which are the same (universal) in the isotropic case at the one-loop level for all models of passively advected scalar field by the velocity field with a Gaussian and non-Gaussian statistics, are different (nonuniversal) in the case with the presence of small-scale anisotropy and they are continuous functions of the anisotropy parameters. The difference is demonstrated by the comparison of the anisotropic rapid-change model [26] with the model studied in the present paper, namely, the model of a passively advected scalar quantity by the turbulent environment given by the corresponding stochastic Navier-Stokes equation (see Figs. 12–21). It must be mentioned once more that all calculations within the present paper were done in the weak anisotropy limit, i.e., we have supposed that the absolute values of the anisotropy parameters are small enough to allow using the linear approximation with respect to them. Without a doubt, for deeper and more adequate comparison of the models, it is necessary to go beyond the weak anisotropy limit calculations. In this respect at least one interesting question remains still open, namely, whether the existence of the regions of anisotropy parameters within the Gaussian models of passive advection [26,34] for which the critical dimension $\Delta[3,1]$ becomes negative and the skewness factor R_3 in Eq. (94) increases going down to the depth of the inertial interval in contrast to the isotropic case and to the case when anisotropy parameters are relatively small is an artifact related to the Gaussian statistics of the velocity field, or it also holds for more realistic models where the statistics of the velocity field is given by the stochastic Navier-Stokes equation. Of course, this question can be answered only when the corresponding models will be compared in the so-called strong anisotropy case, i.e., in the case with no restrictions on anisotropy parameters.

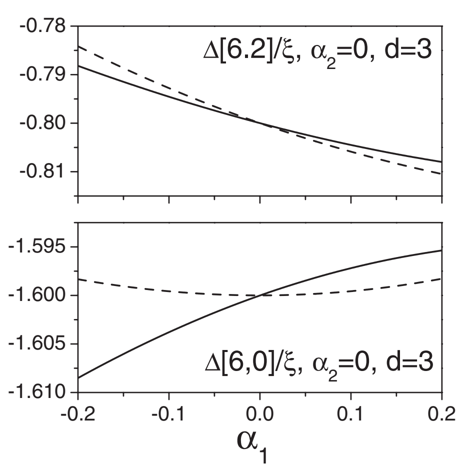

FIG. 18. Dependence of the critical dimension $\Delta[6,p]/\xi$ for $p=0,2$ on the anisotropy parameter α_1 for $\alpha_2=0$ in the weak anisotropy limit and for spatial dimension $d=3$ (for the notation, see the caption of Fig. 12).

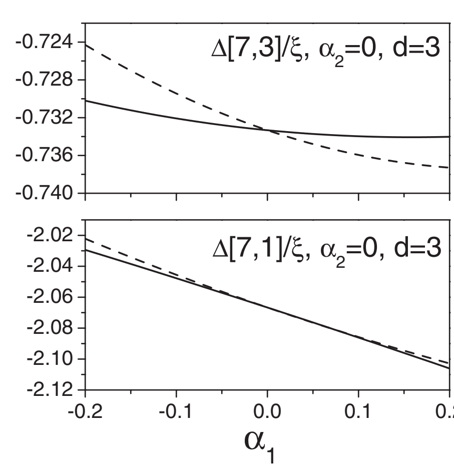

FIG. 20. Dependence of the critical dimension $\Delta[7,p]/\xi$ for $p=1,3$ on the anisotropy parameter α_1 for $\alpha_2=0$ in the weak anisotropy limit and for spatial dimension $d=3$ (for the notation, see the caption of Fig. 12).

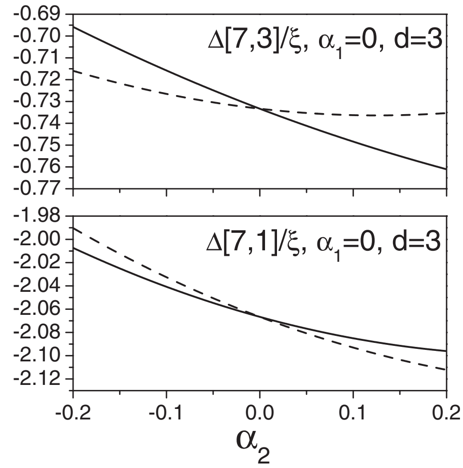

FIG. 21. Dependence of the critical dimension $\Delta[7,p]/\xi$ for $p=1,3$ on the anisotropy parameter α_2 for $\alpha_1=0$ in the weak anisotropy limit and for spatial dimension $d=3$ (for the notation, see the caption of Fig. 12).

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APPENDIX A

The explicit form of the coefficients $g_{j*}, u_{j*}, \chi_{ik*}, \tau_{k*}$ for $i=1,2,3$; $j=0,1,2$,; and $k=1,2$ from Eqs. (48)–(51) is

$$g_{0*} = \frac{8(d+2)}{3(d-1)}\varepsilon,$$

$$g_{1*} = -g_{0*}\frac{3d^4+4d^3+d^2-8d-32}{dM_1},$$

$$g_{2*} = g_{0*}\frac{6d^4-d^3+5d^2-52d-16}{dM_1},$$

$$u_{0*} = \frac{1}{2}\left(-1+\frac{\sqrt{9d+16}}{\sqrt{d}}\right),$$

$$u_{1*} = \frac{16(d+2)N_1}{d^{3/2}M_1M_2M_3}, \quad u_{2*} = \frac{8(d+2)N_2}{d^{3/2}M_1M_2M_3},$$

$$\chi_{11*} = \frac{6d^2+6d-32}{M_1},$$

$$\chi_{12*} = \frac{3d^4+d^3-24d^2+10d+16}{M_1},$$

$$\chi_{21*} = \frac{-8(d^2-3)}{M_1}, \quad \chi_{22*} = \frac{8(3d^2-3d-5)}{M_1},$$

$$\chi_{31*} = 0, \quad \chi_{32*} = 0,$$

$$\tau_{1*} = \frac{N_3}{M_1M_3}, \quad \tau_{2*} = \frac{N_4}{M_1M_3},$$

where

$$M_1 = 3d^4+4d^3-5d^2+10d-56,$$

$$M_2 = 9d+16+\sqrt{d(9d+16)},$$

$$M_3 = d^{3/2}(d+1)+\sqrt{9d+16}(d^2+d-4),$$

$$N_1 = 9d^5-11d^4-102d^3-42d^2+240d+256+\sqrt{d(9d+16)} \times(d^2-2)(d+1)(5d-8),$$

$$N_2 = 81d^5-63d^4-512d^3+104d^2+784d+256 + \sqrt{d(9d+16)}(d+1)\{16+d[24+d(33d-80)]\},$$

$$N_3 = 2\sqrt{9d+16}\{48+d(d+1)[d(d-5)-4]\} + 2\sqrt{d}(d+1)\{32+d[-24+d(9d-13)]\},$$

$$N_4 = \sqrt{9d+16}(64+d[-48+d(d-1)\{6+d[-22 + d(3d+7)]\}])] + \sqrt{d}(d+1)(-32+d\{-72+d[146 + d(3d^2+d-66)]\}),$$

APPENDIX B

The explicit form of the coefficients Q_i with $i=1,2,3$ from Eq. (82) is

$$Q_1 = 2p(p-1)((u+1)\{d^2+5d+4+3[\alpha_2+(d+3)\alpha_1]\} - 3(u+2)[\chi_2+(d+3)\chi_1]-3(d+3)(2u+1)\tau),$$

$$Q_2 = 4(d+4)\{-p(p-1)+(d-1)[(2+d+2p)n+2n(n-1)]\} \times(u+1)-4\alpha_1\{6p(p-1)-(d+1)[(4+d+6p)n + 2n(n-1)]\}(1+u)2\{(d^2-4)p(p-1) + 2[(4+d+2p-4dp)n+6n(n-1)]\}[(1+u)\alpha_2 - (2+u)\chi_2]+4\chi_1\{6p(p-1)-(1+d)[(4+d+6p)n + 2n(n-1)]\}(2+u),$$

$$Q_3 = 8[(4+d+6p)n-2(d-2)n(n-1)][(2+u)\chi_1 - (1+u)\alpha_1+(2u+1)\tau]+4\{[d^3-4(2+p)+2d^2(2+p) + 2d(2p-1)]n+2(d^2-4)n(n-1)\}[(1+u)\alpha_2 - (2+u)\chi_2],$$

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